

Matrices:

Solving Equations with Inverses

Recd #5

$$3x = 6$$

$$\left(\frac{1}{3}\right)(3)x = \left(\frac{1}{3}\right)6$$

↑
inverse

MATRICES.

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

$$\sin A = \frac{1}{2}$$

$$\sin^{-1}(\sin A) = \sin^{-1}\left(\frac{1}{2}\right)$$

$$A = \sin^{-1}\left(\frac{1}{2}\right)$$

$$A = 30^\circ$$

$$A = \pi/3$$



$$A^{-1}A = I$$

EQUATIONS ARE SOLVED BY THE
APPLICATION OF INVERSE.

For example.
Solve

$$5x = 25$$

$$\begin{aligned} 5^{-1}(5x) &= 5^{-1}(25) \\ \frac{1}{5}(5x) &= \frac{1}{5}(25) \\ x &= 5 \end{aligned}$$

$$x^2 = 49 \quad x > 0$$

What is inverse of $()^2$

Apply inverse

$$\begin{aligned} \sqrt{x^2} &= \sqrt{49} \\ x &= 7 \end{aligned}$$

$$\sin x = \frac{1}{2}$$



$$\sin^{-1}(\sin x) = \sin^{-1}\left(\frac{1}{2}\right)$$

$$\begin{aligned} x &= \sin^{-1}\left(\frac{1}{2}\right) \\ x &= 30^\circ \end{aligned}$$

$$10^x = 1000$$

$$\begin{aligned} \log(10^x) &= \log(1000) \\ x &= 3 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Method

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

Need a
recipe

Inverses and Matrix Equations

PW 4-4 p. 74

$$2 \times 2 \ A^{-1}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A)$$

Adjoint =

Transpose of matrix of cofactors

Matrix of cofactors

$$M =$$

TRANSPOSE

Reflection about Diagonal.

Definition of inverse implies

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

The adjoint matrix is the transpose of the cofactor matrix. The cofactor matrix is the matrix of determinants of the minors A_{ij} multiplied by -1^{i+j} . The i,j 'th minor of A is the matrix A without the i 'th column or the j 'th row.

Ex. $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ $A^{-1} = \frac{1}{\begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix}} \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}$

$= \frac{1}{5} \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}$ done.

Check $A^{-1}A$

$$\frac{1}{5} \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} =$$

Inverses and Matrix Equations

PW 4-4 p. 74

$$A =$$

Transpose of matrix of cofactors

Matrix of cofactors

$$M =$$

TRANSPOSE

Reflection about Diagonal.

Definition of inverse implies

Inverses and Matrix Equations

PW 4-4 p. 74

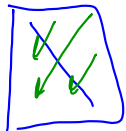
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \frac{1}{\det A} \operatorname{adj}(A)$$

TRANSPOSE
MATRIX OF
COFACTORS

$$A^{-1} = \frac{1}{ad - cb} \operatorname{adj}(A)$$

MATRIX OF COFACTORS

$$M = \begin{bmatrix} + (d) & - (c) \\ - (b) & + (a) \end{bmatrix} \begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

TRANSPOSE  Reflection about Diagonal.

$$M = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

$$M^T = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \operatorname{adj}(A)$$

$$A^{-1} = \frac{1}{ad - cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Def. of Inverse implies

$$AA^{-1} = I \quad A^{-1}A = I$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ identity}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -3 & -2 & 0 \end{pmatrix}$$

Find the inverse of the matrix.

1. $\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 5 \\ 4 & 11 \end{bmatrix}$

3. $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$A^{-1} = \frac{1}{2 \cdot 4 - 3 \cdot 3} \begin{bmatrix} 4 & -3 \\ -3 & 2 \end{bmatrix}$$

$$= \frac{1}{-1} \begin{bmatrix} 4 & -3 \\ -3 & 2 \end{bmatrix} =$$

$$= -1 \begin{bmatrix} 4 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} = A^{-1}$$

Distribute.

matrix

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$3. A^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Solve a Matrix Equation

If $AX = B$ then we use the inverse of A , A^{-1} , to solve.

$$\underbrace{A^{-1}A}X = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

ON THE LEFT!

$$\left(A^{-1}B \neq BA^{-1} \right)$$

unless $A^{-1} = I$
 $B = I$

ASIDE

$$2x = 14$$

INVERSE of 2
is $\frac{1}{2} = 2^{-1}$

$$2^{-1} \cdot 2x = 2^{-1} \cdot 14$$

$$\frac{2}{2}x = \frac{14}{2}$$

$$x = 7.$$

EXAMPLE FIND

4 unknowns

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = X$$

$$\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} X = \begin{bmatrix} 12 & 2 \\ 17 & 3 \end{bmatrix}$$

2x2 2x2 2x2

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 12 \\ 34 \end{pmatrix} = \begin{pmatrix} 12 \\ 34 \end{pmatrix}$$

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

$$X = A^{-1}B$$

$$X = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 12 & 2 \\ 17 & 3 \end{bmatrix}$$

$$\frac{1}{\begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix}} \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 12 & 2 \\ 17 & 3 \end{bmatrix} =$$

$$\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$$

~~1~~

$$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} X = \begin{bmatrix} 12 & 2 \\ 16 & -5 \end{bmatrix}$$

$$\frac{1}{4} \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 12 & 2 \\ 16 & -5 \end{bmatrix} = \begin{bmatrix} 4 & 16 \\ 20 & -12 \end{bmatrix} = X$$

$$X = \begin{bmatrix} 4 & 16 \\ 20 & -12 \end{bmatrix}$$

EXAMPLE FIND 4 unknowns $\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = X$

$$\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} X = \begin{bmatrix} 12 & 2 \\ 17 & 3 \end{bmatrix}$$

$2 \times 2 \quad 2 \times 2 \quad 2 \times 2$

$$AX = B$$

$$A^{-1} = \frac{1}{\begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix}} \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{9-8} \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 12 & 2 \\ 17 & 3 \end{bmatrix}$$

$$X = \begin{bmatrix} 3 \cdot 12 + (-2) \cdot 17 & 3 \cdot 2 + (-2) \cdot 3 \\ -4 \cdot 12 + 3 \cdot 17 & -4 \cdot 2 + 3 \cdot 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$$

check:

$$\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 12 & 2 \\ 17 & 3 \end{bmatrix} \quad \checkmark$$

p.74

Solve the matrix equation.

$$4. \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} X = \begin{bmatrix} 12 & 2 \\ 16 & -5 \end{bmatrix}$$

$$5. \begin{bmatrix} 4 & -6 \\ 3 & -4 \end{bmatrix} X = \begin{bmatrix} 6 & 0 \\ 5 & 1 \end{bmatrix}$$

p.74

Solve the matrix equation.

$$4. \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} X = \begin{bmatrix} 12 & 2 \\ 16 & -5 \end{bmatrix}$$

$$5. \begin{bmatrix} 4 & -6 \\ 3 & -4 \end{bmatrix} X = \begin{bmatrix} 6 & 0 \\ 5 & 1 \end{bmatrix}$$

$$AX = B$$

$$A^{-1} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}^{-1} = \frac{1}{6-2} \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{4} \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 12 & 2 \\ 16 & -5 \end{bmatrix}$$

$$X = \frac{1}{4} \begin{bmatrix} 36-32 & 6+10 \\ -12+32 & -2-10 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4 & 16 \\ 20 & -12 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 4 \\ 5 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} -4 & 6 \\ -3 & 4 \end{bmatrix}$$

$$X = \frac{1}{2} \begin{bmatrix} -4 & 6 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix}$$

Inverses Solve Equations

① $AX = B$

Definition of Inverse

$$\begin{cases} A^{-1}A = I \\ AA^{-1} = I \end{cases}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ e.g. } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Apply inverses to both sides

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

↑

multiply from left.

Inverses Solve Equations

$$A^{-1} = \frac{1}{ad-cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Def. of Inverse implies

$$AA^{-1} = I \quad A^{-1}A = I$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ identib.}$$

EQUATION

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

Example

Use matrices to solve the linear system. $5x + 2y = 3$

$4x + 2y = 4$

(1) WRITE EQUATIONS AS MATRICES.

(2) FIND INVERSE OF A.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$2 = 5 \cdot 2 - 2 \cdot 4$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 2 & -2 \\ -4 & 5 \end{bmatrix}$$

3. MULTIPLY

$$X = A^{-1}B \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & -2 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 \cdot 3 + (-2) \cdot 4 \\ (-4) \cdot 3 + 5 \cdot 4 \end{bmatrix}$$

$2 \times 2 \quad 2 \times 1 \quad 2 \times 1$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -2 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$(-1, 4)$$

$$A X = B$$

$$\begin{bmatrix} 5 & 2 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

matrix of coefficients

vector of variables

vector of constants.

4-5 Solve Systems with Inverse Matrices, page 76

Example: Use an inverse to solve

$$2x + 3y = 1$$

$$-3x - 6y = 5$$

$$\begin{bmatrix} 2 & 3 \\ -3 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

1. EQUATIONS

$$2. \text{ INVERSE } A^{-1} = \frac{1}{-3} \begin{bmatrix} -6 & -3 \\ 3 & 2 \end{bmatrix}$$

$$3. \text{ MULTIPLY } \begin{bmatrix} x \\ y \end{bmatrix} = \frac{-1}{3} \begin{bmatrix} -6 & -3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} -6 \cdot 1 + -3 \cdot 5 \\ 3 \cdot 1 + 2 \cdot 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} -21 \\ 13 \end{bmatrix} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -13/3 \end{bmatrix}$$

4-5 Solve Systems with Inverse Matrices, page 76

Use matrices to solve the linear system. $5x + 2y = 3$

$4x + 2y = 4$

$$\begin{bmatrix} 5 & 2 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$A X = B$$

$$2 \times 2 \quad 2 \times 1 \quad 2 \times 1.$$

$$X = A^{-1} B$$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 2 & -2 \\ -4 & 5 \end{bmatrix}$$

$$X = \frac{1}{2} \begin{bmatrix} 2 & -2 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 6-8 \\ -12+20 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

$$X = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$\begin{cases} x = -1 \\ y = 4 \end{cases}$$

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Solve for x and y

1.

$$\begin{aligned} 2x - y &= 1 \\ -3x + 2y &= 0 \end{aligned}$$

2. $3x + 4y = -4$

$4x + 5y = -7$

$$\begin{aligned} \begin{matrix} 1 & 2x - y = 1 \\ 0 & -3x + 2y = 0 \end{matrix} & \quad A^{-1} = \frac{1}{(2)(2) - (-3)(1)} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} \\ \begin{bmatrix} 5 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \quad X = A^{-1}B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \end{aligned}$$

3. $6x - 5y = 3$

$3x - 2y = 3$

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M19.

Solve for x and y

1.

$$\begin{aligned} 2x - y &= 1 \\ -3x + 2y &= 0 \end{aligned}$$

$$\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A X = B$$

$$2 \times 2 \quad 2 \times 1 \quad 2 \times 1.$$

2. $3x + 4y = -4$

$4x + 5y = -7$

(1) WRITING MATRIX EQ.

$$\begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} 5 & -4 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} -4 \\ -7 \end{bmatrix} = + \begin{bmatrix} 8 \\ -5 \end{bmatrix}$$

3. $6x - 5y = 3$

$3x - 2y = 3$

$$\begin{aligned} 1 \cdot x - y &= 1 \\ 2x - 3y &= 0 \end{aligned} \quad A^{-1} = \frac{1}{2(-1) - (-2)} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad X = A^{-1}B = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$